Natural Frequencies of Euler-Bernoulli Beam with Open Cracks on Elastic Foundations

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A study of the natural vibrations of beam resting on elastic foundation with finite number of transverse open cracks is presented. Frequency equations are derived for beams with different end restraints. Euler-Bernoulli beam on Winkler foundation and Euler-Bernoulli beam on Pasternak foundation are investigated. The cracks are modeled by massless substitute spring. The effects of the crack location, size and its number and the foundation constants, on the natural frequencies of the beam, are investigated.

Key Words : Euler-Bernoulli Beam, Elastic Foundation, Crack, Pasternak Foundation, Winkler Foundation

1. Introduction

The analysis of beams on an elastic foundation is developed on the assumption that the reaction forces of the foundation are proportional to the deflection of the beam. This assumption was introduced by E. Winkler (Hetenyi, 1946). Pasternak proposed a foundation model consisting of a Winkler-type foundation with shear interactions (Rades, 1970).

Dynamics and stability of the Winkler-type foundation model have been thoroughly investigated by both approximate methods (De Rosa, 1989) and exact approaches (Farghaly and Zeid, 1995 ; Maurizi et al., 1988). Some finite element models for the static analysis of Euler-Bernoulli beam resting on a Winkler-type foundation have been given by Razaqpur and Shah (1991). The same beam on a Pasternak two-parameter foundation has been analyzed in an exact way by Valsangkar and Pradhanang (1988), and the corresponding Timoshenko beam has been studied by Rosa (1995).

In order to investigate the effects of damage presented in the structure, several studies were introduced through a simple reduction of the stiffness in the mathematical model (Yuen, 1985 ; Joshi and Madhusudhan, 1991). Christides et al. developed a cracked Euler-Bernoulli beam theory by deriving the differential equation and related boundary conditions for a uniform beam with one or two pairs of symmetric cracks. To deal with the effects of cracks on the eigenparameters, in some articles the beam was subdivided into several beams, separated one another by cracks, which were modeled by massless rotational spring

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(Narkis, 1994; Ostachowicz and Krawczuk, 1991).

The first purpose of this paper analyzes the vibration of the Euler-Bernoulli beam with the crack on elastic foundation like the Winkler and the Pasternak foundations. Next, the results of the Euler-Bernoulli beam on the Winkler foundation is compared with the results of the Euler-Bernoulli beam on the Pasternak foundation on various combinations. Last, the effect of the location of the crack and the depth of the crack is investigated.

2. Dynamic Analysis

2.1 Euler-Bernoulli beam on Winkler foundation

A beam with a crack on Winkler foundation is shown in Fig. 1. The crack is located at point x_1 illustrated in Fig. 1. The beam is assumed to be composed of two segments connected by massless substitute spring at crack location. The equation of motion governing the flexural vibration of a uniform rectangular beam is

$$EIy_i^{(IV)}+\rho Ay_i^{(II)}+k_f y_i=0, \quad x_{i-1}\le x\le x_i,\ i=1,2\ (x_0=0,\ x_2=L) \tag{1}$$

where E is the Young's modulus, I is the moment of inertia of the cross-section, ρ is the material density and k_f is the Winkler foundation modulus.

Assuming a steady-state solution

$$y_i(x,\ t)=Y_i(x)\,e^{j\omega t} \quad i=1,\ 2 \tag{2}$$

Substituting Eq. (2) into Eq. (1),

$$\frac{d^4Y_i}{dX^4}-(\lambda^4-K)\,Y_i=0 \quad X_{i-1}\le X\le X_1,\ i=1,\ 2 \tag{3}$$

where

$$X=\frac{x}{L},\ X_i=\frac{x_i}{L},\ \lambda^4=\frac{mL^4\omega^2}{EI} \text{ and } K=\frac{k_fL^4}{EI}.$$

The general solutions of Eq. (3) are

$$Y_i(X)=A_i\cos(\alpha X)+B_i\sin(\alpha X)+C_i\cosh(\alpha X)+D_i\sinh(\alpha X) \tag{4}$$

where A_i, B_i, C_i, D_i are constants ($i=1$, 2) and $\alpha=\lambda^4-K$.

2.2 Euler-Bernoulli beam on Pasternak foundation

The differential equation of the transverse vibration of a flexibly supported Euler-Bernoulli beam on Pasternak foundation is

$$EIy_i^{(IV)}+(\rho A-G_0)\,y_i^{(II)}+k_f y_i=0, \quad x_{i-1}\le x\le x_i,\ i=1,\ 2\ (x_0=0,\ x_2=L) \tag{5}$$

where G_0 is the shear modulus of foundation. A steady-state solution is also assumed as Eq. (2).

Substituting Eq. (2) into Eq. (5),

$$\frac{d^4Y_i}{dX^4}-s^2\frac{d^2Y_i}{dX^2}-(\lambda^4-K)\,Y_i=0, \quad X_{i-1}\le X\le X_i,\ i=1,\ 2 \tag{6}$$

where $s^2=\frac{G_0L^2}{EI}$.

The general solutions of Eq. (6) are

$$Y_i(X)=A_i\cos(\alpha X)+B_i\sin(\alpha X)+C_i\cosh(\beta X)+D_i\sinh(\beta X) \tag{7}$$

where A_i, B_i, C_i, D_i are constants ($i=1$, 2)

$$\alpha=\frac{1}{\sqrt{2}}\left[\{s^4+4(\lambda^4-K)\}^{\frac{1}{2}}-s^2\right]^{\frac{1}{2}}$$

$$\beta=\frac{1}{\sqrt{2}}\left[\{s^4+4(\lambda^4-K)\}^{\frac{1}{2}}+s^2\right]^{\frac{1}{2}}$$

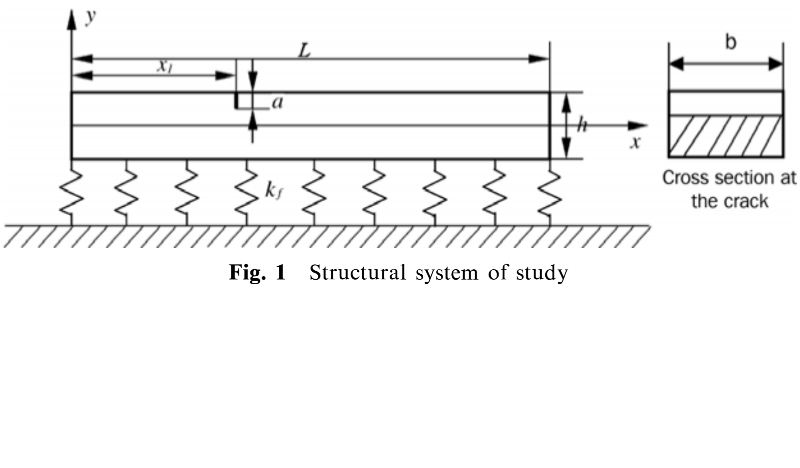

Fig. 1 Structural system of study

2.3 Crack compliance

For compatibility of displacements, moments and shear forces of both segments at the crack, the angular displacement between the two segments can be related to the following

$$Y_1(l_1) = Y_2(l_2) \tag{8}$$

$$Y_1^{(\mathrm{II})}(l_1) = Y_2^{(\mathrm{II})}(l_2) \tag{9}$$

$$Y_1^{(\mathrm{III})}(l_1) = Y_2^{(\mathrm{III})}(l_2) \tag{10}$$

$$Y_2^{(\mathrm{I})}(l_2) - Y_1^{(\mathrm{I})}(l_1) = c\,Y_1^{(\mathrm{II})}(l_1) \tag{11}$$

where c is the flexibilities of the rotational springs which are functions of the crack extent and beam width. The c for one-sided crack can be expressed as (Narkis, 1994).

$$c=5.346\cdot h\cdot f(\xi) \tag{12}$$

where h is the height of the cross-section of the beam, $\xi=a/h$, where a is the depth of the crack and

$$\begin{aligned} f(\xi) = {} & 1.8624\xi^2-3.95\xi^3+16.375\xi^4 \\ & -37.226\xi^5+76.81\xi^6-126.9\xi^7 \\ & +172\xi^8-143.9\xi^9+66.56\xi^{10} \end{aligned} \tag{13}$$

2.4 Natural frequencies of a cracked beam on foundation

The boundary conditions for the beams with different end restraints are as follows

fixed: $Y=0,\ Y^{(\mathrm{I})}=0$ (14, 15)

free: $Y^{(\mathrm{II})}=0,\ Y^{(\mathrm{III})}=0$ (16, 17)

supported: $Y=0,\ Y^{(\mathrm{II})}=0$ (18, 19)

If Eqs. (4) and (7) is inserted into Eqs. (8) ~ (11) and boundary conditions, 8 by 8 matrix equations is obtained. For two cracks, 12 by 12 matrix equation is obtained. Natural frequencies are calculated by imposing zero value on the determinant of this coefficient matrix.

3. Numerical Analysis and Discussions

The Mathematica® version 4.0 has been used for all the computational processes in this paper. The cantilever beam under analysis has the following properties: length $L=10$ m, Young's modulus $E=2.068\times10^{11}\ \mathrm{N/m^2}$, material density $\rho=7850\ \mathrm{kg/m^3}$, rectangular cross section with width $b=0.25$ m and height $h=0.25$ m.

Tables 1 and 2 respectively show the first natural frequencies of fixed-fixed beam and supported-supported beam for the location and the depth of a crack, K=10 and s=5. The results of the Euler-Bernoulli beam on the Pasternak foundation are bigger than the results of the Winkler foundation. It shows that the frequencies of the beam resting on a Pasternak foundation are higher than those of the beam on a Winkler foundation.

Table 3 shows the first three natural frequencies of fixed-fixed beam for the depth of a crack, K=10 and s=5.

Table 4 represents the increment(%) of the first natural frequencies of the beam on Pasternak foundation compared to those on Winkler foundation on combination of fixed, simple-supported and free end. Except the results of the fixed-free beam, all results of the Euler-Bernoulli beam on

Table 1 A comparison between the first natural frequencies (Hz) for fixed-fixed beam on Winkler and those on Pasternak foundation with respect to the crack ratio for K=10 and s=5

a/h \ X_1	1/8		1/4		1/2	
	Winkler	Pasternak	Winkler	Pasternak	Winkler	Pasternak
0.02	83.6699	105.669	83.6964	105.669	83.6401	105.578
0.04	83.5904	105.641	83.6920	105.638	83.4755	105.292
0.2	81.7961	105.001	83.5809	104.843	79.4647	98.1936
0.4	78.7395	103.889	83.3187	102.774	71.0004	82.4814
0.5	77.4239	103.403	83.1621	101.404	66.5043	73.6336

Table 2 A comparison between the first natural frequencies (Hz) for supported-supported beam on Winkler and those on Pasternak foundation with respect to the crack ratio for K=10 and s=5

a/h \ X_1	1/8		1/4		1/2	
	Winkler	Pasternak	Winkler	Pasternak	Winkler	Pasternak
0.02	38.3844	69.6986	38.3727	69.6757	38.3561	69.6433
0.04	38.3705	69.6714	38.3251	69.5831	38.2614	69.4590
0.2	37.9790	68.8996	37.0428	67.0450	35.8669	64.7705
0.4	36.6236	66.0965	33.2729	59.2578	30.1536	53.3985
0.5	35.2968	64.6815	30.3605	55.3642	26.6241	46.2470

Table 3 A comparison between the first three natural frequencies (Hz) for fixed-fixed beam on Winkler and those on Pasternak foundation with respect to the crack ratio for $X_1=1/8$, K=10 and s=5

a/h	First		Second		Third	
	Winkler	Pasternak	Winkler	Pasternak	Winkler	Pasternak
0.02	83.6699	105.669	228.745	260.847	447.938	484.259
0.04	83.5904	105.641	228.742	260.849	447.766	484.011
0.2	81.7961	105.001	228.676	260.911	443.708	478.022
0.4	78.7395	103.889	228.564	261.024	436.040	466.031
0.5	77.4239	103.403	228.515	261.077	432.468	460.198

Table 4 The increment (%) of the first natural frequencies of the beam on Pasternak foundation compared to those on Winkler foundation

X_1	1/8	1/4	1/2
F-F and Fr-Fr	29.3	24.6	20.6
S-S	81.7	80.9	78.9
F-Fr	−22.5	−23.6	−26.1
F-S	51.9	45.9	39.8
S-Fr	46.4	46.2	45.3

*F : fixed, S : simply-supported, Fr : free

the Pasternak foundation are higher.

Figures 2 and 3 respectively represent the first and the second natural frequencies on the Pasternak foundation for fixed-fixed beam with respect to K=10 and s=5. It shows that the variations of natural frequencies are sensitive for the location of the crack.

Figure 4 represents the first natural frequencies on the Winkler foundation for fixed-fixed boundary condition with respect to K=10 and $X_2=7/8$.

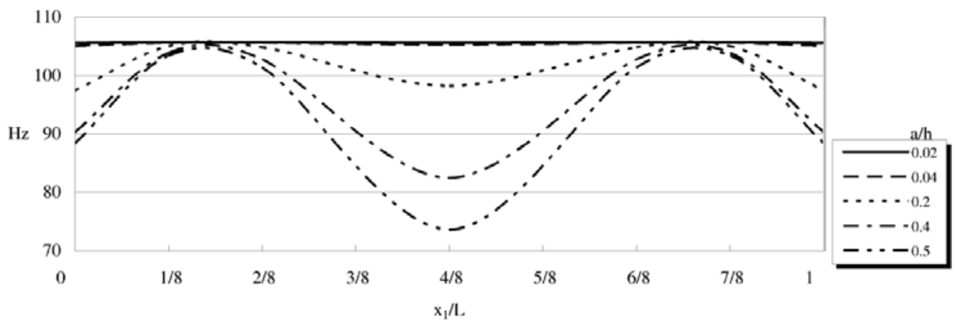

Fig. 2 The variations of the first natural frequencies for the fixed-fixed beam on Pasternak foundation due to crack position $X_1=x_1/L$ for K=10 and s=5

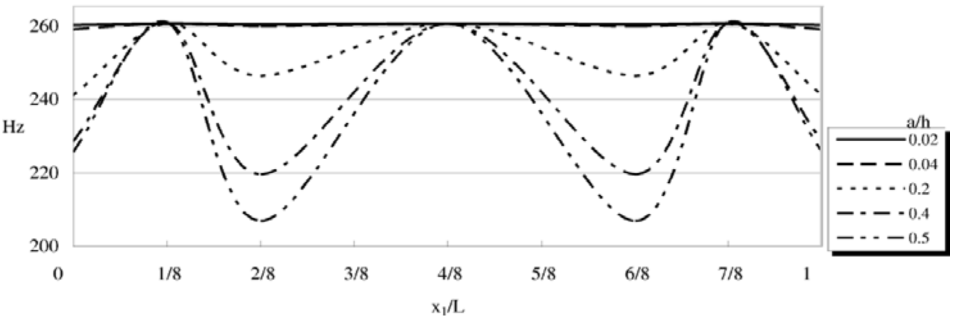

Fig. 3 The variations of the second natural frequencies for the fixed-fixed beam on Pasternak foundation due to crack position $X_1 = x_1/L$ for $K = 10$ and $s = 5$

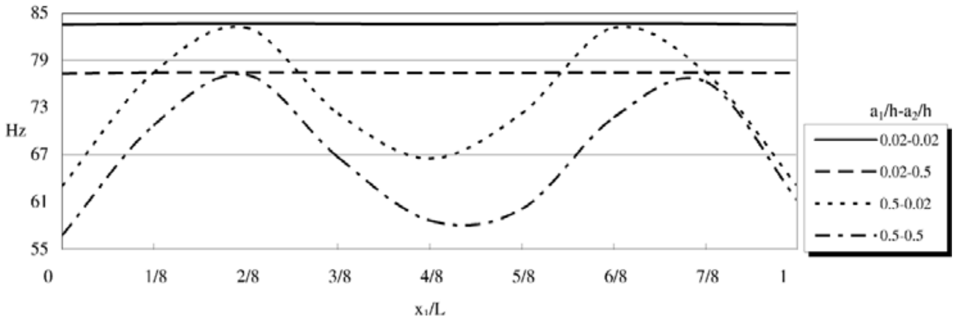

Fig. 4 The first natural frequencies of the fixed-fixed beam with two cracks on Winkler foundation for $K = 10$ and $X_2 = 7/8$

4. Conclusions

In this paper, the natural frequencies of the cracked beam resting on elastic foundations are investigated.

(1) Except the results of the fixed-fixed boundary condition, the frequencies of the beam resting on a Pasternak foundation are higher than those of the beam on a Winkler foundation.

(2) The location of the crack affects to the changes in the frequencies of the natural vibrations significantly.

(3) As the depth of the crack increases, the frequencies decrease significantly.

Acknowledgments

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